When a value judgement using the technique of unobserved DMUs can be changed by weight restrictions

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Abstract *Data Envelopment Analysis - DEA, developed originally by Charnes et al. (1978), is a method that uses linear programming for the evaluation of comparative efficiencies of the Decision-Making Units (DMUs). Classic multipliers DEA models allow each weight to assume any positive value, in order to maximize a DMU's efficiency, thus often resulting excessive weight variability and implausible weight values, under DM's viewpoint. This has led to the development of DEA models that incorporate weight restrictions, reflecting expert judgment, only the establishement of bounds is not a straightforward procedure. As an alternative, Thanassoulis, E. and Allen, R. (1998) have demonstrated that it is possible to find an unobserved DMU that is a substitute for a set of Weight Restrictions in DEA. However, they have not established the necessary conditions for an unobserved DMU to have an equivalent set of weight restrictions. This paper aims to provide a solution for this issue, presenting the necessary conditions to ensure this equivalence.*

Key words: Data Envelopment Analysis, Decision-maker preferences, Unobserved DMU

1. Introduction

DEA evaluation is calculated by the solution of a Linear Programming Problem (LPP) whose multipliers version attributes weights to the Input and Output data, in order to maximize the efficiency. Due to the diffusion of its application, weight restrictions had to be included in the model so as to better reflect the specialists' value judgments in each case.

Thanassoulis, E. and Allen, R. (1998), demonstrated that it is possible to substitute a set of weight restrictions for an Artificial (or unobserved) DMU, preserving the same efficiency of the original DMUs.

Allen and Thanassoulis, managed to demonstrate how to find the coordinates of this artificial DMU, starting from a set of original DMUs with known weight restrictions. However, the conditions for the complete equivalence were not established, to guarantee the existence of a set of weight restrictions equivalent to an artificial DMU.

The objective of this paper is to propose a solution to this problem, presenting the necessary conditions to ensure this equivalence.

The section 2 of this text presents a short review of weight restrictions theory. Section 3 shows the Thanassoulis & Allen proposal aimming to find atificial DMUs that are equivalent to a weight restrictions set. In the section 4, we discuss the general question of where to locate an artificial DMU in order to assure the existence of an equivalent set of weight restrictions. The section 5 presents a theorem that states the conditions of equivalence. Section 6 concludes.

2. Incorporation of the specialist's value judgement through weight restrictions

Considering the multipliers DEA model, a set of weights represents a system of relative values for each Input and Output, which gives the largest possible efficiency level for a certain DMU under analysis.

In classic DEA models there is no way of incorporating the preferences of the decision-maker or value judgement of the specialists.

The flexibility of the weights is one great advantage provided by the DEA methodology. However, this total freedom of weight attribution can often yield unacceptble weights under de decision makers' point of view.

The incorporation of the specialist's value judgment, or preference of the decision-makers, can provide adequate ranges for the weights, implemented through the inclusion of weight restrictions. Allen, Athanassopoulos, Dyson and Thanassoulis (1997) classified weight restriction methods into three cathegories:

2.1 Direct imposition of the weight restrictions;
2.2 Adjustments in the levels of observed inputs and outputs;
2.3 Virtual Input and Output restrictions.

3. Equivalence between weight restrictions and an artificial DMU

Roll and Golany (1991) initially noted that each DEA positively restricted weight was equivalent to an unobserved DMU included in the original set of DMUs.

Thanassoulis and Allen (1998) generalized this result for the case of multiple inputs and/or outputs, both for CRS and VRS models. They also developed an alternative method, substituting an artificial DMU for a set of weight restrictions. The efficiencies obtained with this artificial DMU were shown to be identical to the ones obtained by the weights restricted model.

In order to demonstrate this assumption, Allen and Thanassoulis started from the basic DEA/CRS model, with AR I and AR II restrictions (r1, r2 and r3), presented in the equations (1) to create two different sets of unobserved DMUs.

The first set, FSUD (Full Set of Unobserved DMUs), is composed of unobserved DMUjt , one for each DMUj, j=1,...,J, presenting output yrjt, with r=1,...,R and input xijt, i=1,...,I, where:

$$y_{rjt} = \frac{y_{rj}}{h_j^*} \quad \textit{and} \quad x_{ijt} = x_{ij} \quad \forall jt = j \tag{4}$$

If (1) has a feasible solution, then $h'_{j0} = h^*_{j0}$ where:

$$\begin{aligned} h'_{j0} &= \text{Max} \sum_{r=1}^{R} \delta_r y_{rj0} \\ &\sum_{i=1}^{I} \gamma_i x_{ij0} = 1 \\ &\sum_{r=1}^{R} \delta_r y_{rj} - \sum_{i=1}^{I} \gamma_i x_{ij} \le 0, \quad j = 1,...j...,J \\ &\sum_{r=1}^{R} \delta_r y_{rjt} - \sum_{i=1}^{I} \gamma_i x_{ijt} \le 0, \quad j_t = 1,..,J \\ &\delta_r, \gamma_i \ge \varepsilon \quad \forall r,i \end{aligned} \tag{5}$$

Thus, the same results obtained in (1), only considering the set of restrictions AR I and AR II (r1, r2 and r3) , are obtained in (5), without weight restrictions and with the inclusion of the unobserved DMU_{jt} , j=1,..., J. In the case of the DEA/CRS model, we could alternatively change x_{ijt} and keep y_{rjt} constant, as shown in (6).

$$y_{rjt} = y_{rj} \quad \text{and} \quad x_{ijt} = x_{ij}.h_j^* \quad \forall jt = j \tag{6}$$

In case of hj*=1, the non-observed DMU of the FSUD group consists of the very observed DMU, which is a redundant result Then, a subset of FSUD can be created, namely RSUD (Reduced Set of Unobserved DMUs), composed of unobserved DMUs that are not copies of the originals.

Conforming to Thanassoulis and Allen (1998) RSUD also excludes unobserved DMUs whose vectors of Input/Output are dominated by any convex linear combinations of the Input/Output vectors of other observed DMUs. The construction of the RSUD set can be easily obtained by eliminating doubled DMUs and making use of the super-efficiency model (see Andersen and Petersen, 1993) to eliminate DMUs with vectors I/O that are convex linear combination of the others.

The same reasoning is made for the DEA/VRS models, the only difference being that, in this case, the inclusion of unobserved DMUs for simulating the set of restrictions will be done trough unobserved DMUs defined by (4) and (6) when working, respectively, with expansion of outputs or contraction of inputs.

To exemplify this theorical achievements, let us take the set of DMUs presented in Table 1, with two Inputs X1 and X2 and one constant Output Y. This table shows the results of the multipliers DEA/CRS model, calculated without weight restrictions. Then we imposed a restriction to the weight of Input X1 to be more than or equal to the weight of Input X2 ($v1 \geq v2$) , whose results are shown in the same table.

Table 1- Efficiencies and weights calculated by DEA/CRS.

				Without weight restrictions				With weight restrictions ($v_1 \geq v_2$)			
DMU	X1	X2	Y	h %	v1	v2	u	h %	v1	v2	u
A	1.0	4.0	2.0	100	1	0	0.5	100	1	0	0.5
B	1.0	3.0	2.0	100	0.4	0.2	0.5	100	0.40	0.20	0.5
C	3.0	1.0	2.0	100	0.25	0.25	0.5	100	0.25	0.25	0.5
D	4.0	1.0	2.0	100	0	1	0.5	80.0	0.20	0.20	0.5
E	3.5	2.2	2.0	70.18	0.175	0.175	0.5	70.18	0.175	0.175	0.5
F	1.6	2.8	2.0	90.91	0.227	0.227	0.5	90.91	0.227	0.227	0.5

If we look at the Table 1, we can see that the only DMU whose weights do not satisfy the imposed restriction is DMU **D**. In order to satisfy this requirement, we want to include an Artificial DMU **W** that would alter the section of the frontier where DMU **D** is being projected.

According to equations (6), taking hD=0.80 from Table 1, the new Artificial DMUW will have:

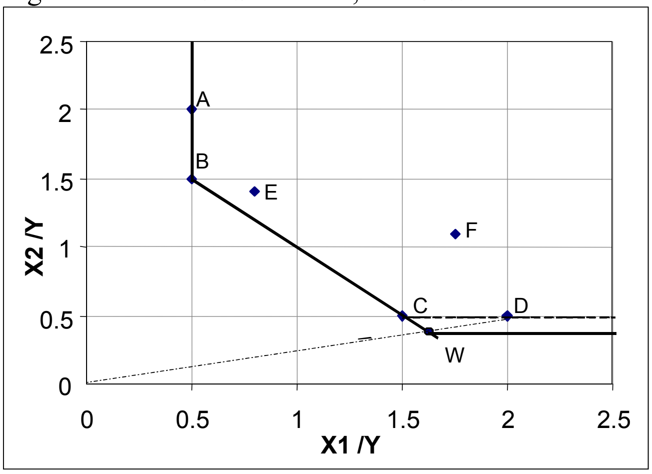
$$X_{W1} = X_{D1} * 0.80 = 3.2 \qquad X_{W2} = X_{D2} * 0.80 = 0.8 \qquad Y_W = Y_D = 2.0$$

Table 2 shows the results for the basic DEA/CRS model calculated with the included DMU W. Figure 1 depicts the old and the new frontier. So, we can verify that the inclusion of the Artificial DMU "W" provided the same indexes of efficiencies as those obtained by calculating DEA with weight restrictions.

Table 2 - DEA without weight restrictions

DMU	h%	v1	v2	u
A	100	1	0	0.5
B	100	0.40	0.20	0.5
C	100	0.25	0.25	0.5
D	80.0	0.20	0.20	0.5
E	70.18	0.175	0.175	0.5
F	90.91	0.227	0.227	0.5
W	100	0	1.25	0.5

Figure 1 - DEA/CRS Frontier, DMU W is included



4. When there is no weight restrictions equivalent to an artificial DMU

Thanassoulis, E. and Allen, R. (1998) have demonstrated that it is possible to find an unobserved DMU that is a substitute for a set of Weight Restrictions in DEA. However, can we affirm that every additional artificial DMU will correspond to a set of weight restrictions? Are there specific conditions to assure this equivalence?

In order to simplify the next presentations, equivalence between an Artificial DMU (AD) and Weight Restrictions (WR) will be defined simply as ***AD-WR Equivalence***, whenever it is possible.

We will start this discussion using the data set from Table 1 and we suppose an Artificial DMU, which is located at an arbitrary point W (X_1/Y=0,7 ; X_2/Y=0,7) as showed in Figure 2. The

Artificial DMU W expanded the frontier and created two new Pareto Efficient (PE) faces, so that the projections of DMUs **E** and **F**, formerly on the hyperplan $\overline{BC}$ (in this case, a straight line segment), are transposed, respectively, to the PE hyperplans $\overline{WB}$ and $\overline{WC}$.

From the results in Table 3 we see that, after the inclusion of W, the weights of Inputs 1 and 2 of **E** and **F**, vary in the opposite direction, that is, the weights V_{1E} decreased and V_{2E} increased, while, for **F**, V_{1F} increased and V_{2F} decreased. It turns out impossible the existence of a single set of weight restrictions that is equivalent to the inclusion of DMU **W**. By analogous reasoning we can conclude that no point in the demarcated region in Figure 3 exhibit *AD-WR Equivalence.*

Figure 2- DEA frontier with inclusion of DMU **W**

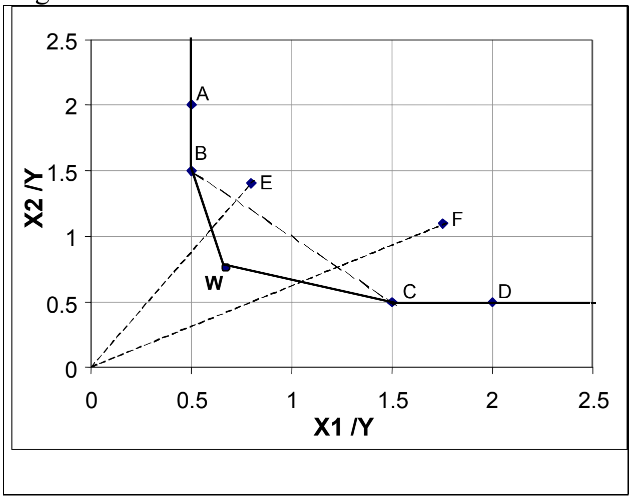
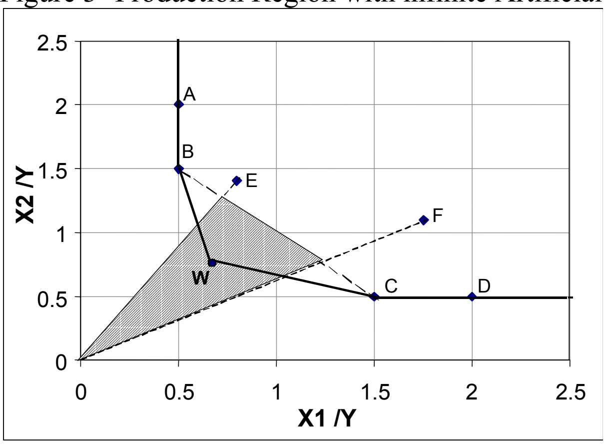


Table 3 : Efficiencies and weights, without weight restrictions (with DMU **W**)

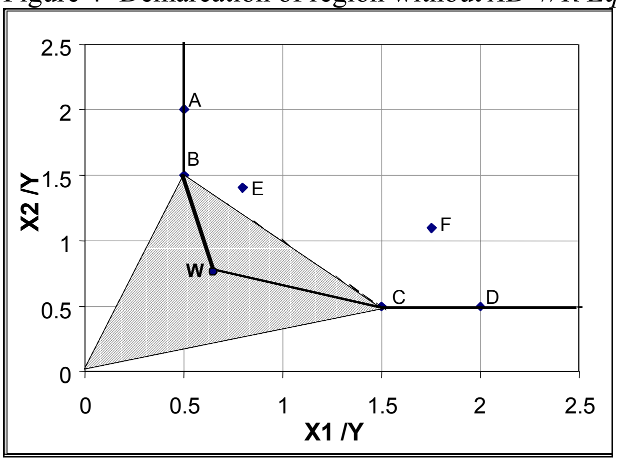
DMU	h%	v1	v2	u
A	100	1	0	0.5
B	100	1	0	0.5
D	100	0	1	0.5
D	100	0	1	0.5
E	56.91	0.081	0.325	0.5
F	76.09	0.435	0.109	0.5
W	100	0.5	0.5	0.5

Figure 3- Production Region with infinite Artificial DMUs without ***AD-WR Equivalence***



Using the same reasoning, we see that projections of the DMUs B and C prevent the *AD-WR Equivalence* for any arificial DMU located inside the cone demarcated in Figure 4 .

Figure 4- Demarcation of region without *AD-WR Equivalence*



Regarding the example presented in Table 1, if the artificial DMU were placed in a location such that one only PE face were produced, there would be AD-WR equivalence. This can be observed in Figures 5 and 6, which show the artificial DMUs W_1 and W_2, respectively.

It is worthwhile recalling that, in Figures 3, the supressed face $\overline{BC}$ can be expressed as a Strict Convex Linear Combination (SCLC) of the new faces produced by the Artificial DMU W. This implies the resulting non-equivalence. In the cases presented in Figures 5 and 6, the supressed faces cannot be expressed as SCLC of the new ones, given that only one PE face was produced.

Figura 5: Artificial DMU W1

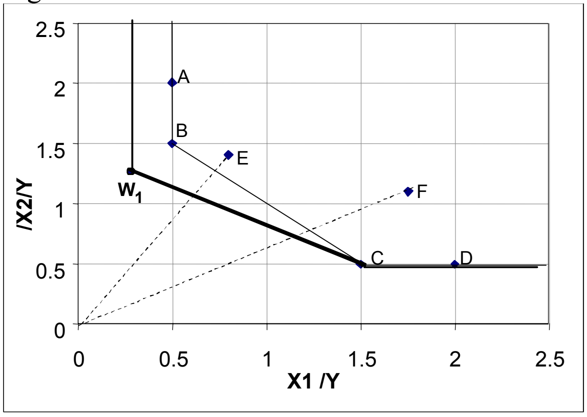
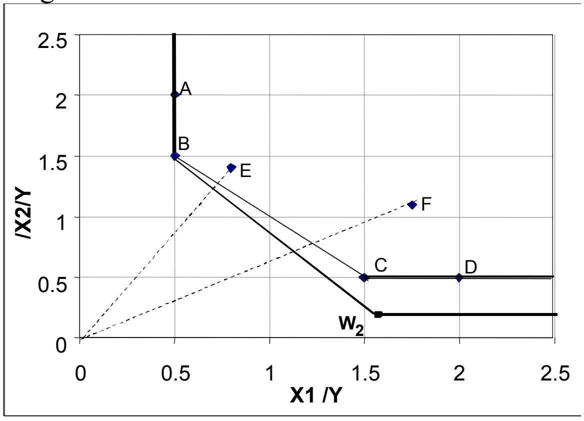


Figura 6: Artificial DMU W2

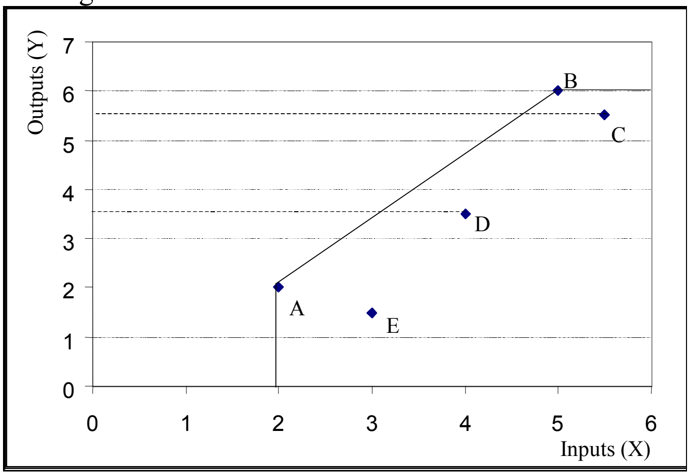


In order to make the verification of AD-WR Equivalence in case of variable returns to scale (DEA/VRS), we could take the set presented in Table 4, with DEA/VRS frontier according to Figure 7.

Table 4 - Set of 1 Input and 1 Output DMUs

DMU	X	Y
A	2,0	2,0
B	5,0	6,0
C	5,5	5,5
D	4,0	3,5
E	3,0	1.5

Figure 7- DEA/VRS frontier of the original DMUs



5. Conditions of equivalence

As we could observe, it is possible to substitute an artificial DMU strategically placed among the original DMUs for a set of weight restrictions in DEA. We could also verify that an artificial DMU will not always be equivalent to a set of weight restrictions. In line with the terminology introduced by Thompson et al (1990), we shall develop the conditions of equivalence, concerning Assurance Regions Type I (ARI) and Assurance Regions Type II (ARII).

It is possible to determine a formulation of conditional rules that ensure this equivalence from the following theorem:

5.1. Definitions and Theorem 1

Definition of DEAj : The DEA multiplier restriction (WR) problem formulated and solved for one only observed DMUj, instead of every DMU will be christened DEAj. This definition applies both to basic and weight restricted models

Definition of DEA-AD and DEA-WR : Let us assume DEA-AD as a DEA basic model without weight restrictions, adding one artificial DMU W, and DEA-WR as the DEA model comprising the original set of DMUs and aditional weight restrictions.

Theorem 1: Consider an artificial DMU added to a basic DEAj model CRS or VRS, such that the efficciency of DMUj is altered. It is allways possible to substitute this DMU for a set of weight restrictions such that the resulting DEAj WR model is AD-WR equivalent.

We will show that, given any artificial DMU located outside the frontier, it is allways possible to find a set of weight restrictions that is AD-WR equivalent, considering a DEAj model for a specific DMUj.

5.2 Theorems 2 and 3

Theorem 2: Given an artificial DMU W added to a basic DEA model CRS or VRS. If any multiplier of at least one supressed face can be expressed as a Strict Convex Linear Combination (SCLC) of the correspondent multipliers of the new PE faces produced by the DMU W, then AD-WR equivalence will not hold.

The bidimensional AR1 and AR2 cases will be demonstrated. Generalisation would result from a straightforward reasoning.

Theorem 3: Given an artificial DMU W added to a basic DEA model CRS or VRS, there will be ***AD-WR*** Equivalence iff this DMU W produces one only Pareto Efficient (PE) face.

Considering the Theorem 1, there will be a weight restricted DEAj program that is AD-WR equivalent to any included artificial DMU. As the resolution of DEA model requires the solution of DEAj , j = 1, ..., J, given a unique set of constraints, AD-WR equivalence will hold for the DEA model iff the added weight restrictions are not in conflict. Considering the developments in Theorem 2 we must show that if one only face is produced, then a conflict is not possible.

i) If there are two or more PE new faces, then it is always possible to have a multiplier of one supressed face that can be expressed as a Strict Convex Linear Combination (SCLC) of the correspondent multipliers of the new PE faces produced by the DMU W. From Theorem 2 follows that there is no AD-WR Equivalence.

ii) If only one PE face is produced, then it is impossible to represent any original face as a SCLC of this unique PE face.

In the sequence, we include an improvement for the concept of AD-WR Equivalence, that takes in consideration the relevance of the efficiencies measured in relation to Pareto Inefficient portions of the frontier.

6. Conclusion

This article contributes for the use of artificial DMUs as proposed by Allen and Thanassoulis, showing that an artificial DMU will not always correspond to a set of weight restrictions in DEA.

Moreover, it presents a theorem that formalizes the necessary conditions for this equivalence to hold.

This development is an important contribution for analysis and incorporation of value judgment of the specialists.

7. References

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